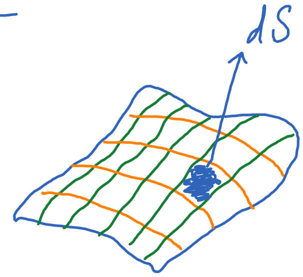
Lecture 20

Monday, June 13, 2022 8:25 AM

* Prayer
* Spiritual thought

Surface integral of a scalar function

$$\iint_S f\, dS$$



$S: r(u,v)$

$(u,v) \in R$

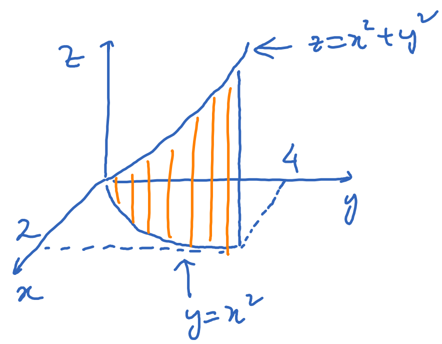
$$dS = |r_u\, du \times r_v\, dv|$$

$$= |r_u \times r_v| \underbrace{du\,dv}_{dA}$$

Therefore,

$$\iint_S f\, dS = \iint_R f(x(u,v), y(u,v), z(u,v))\, |r_u \times r_v|\, dA$$

Ex:



$f(x,y,z) = x$: mass density per unit area

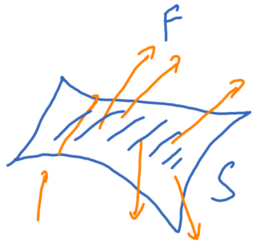
Total mass = ?

$$\begin{cases} x = t \\ y = t^2 \\ z = z \end{cases} \longrightarrow r(t,z) = (t, t^2, z) \rightsquigarrow \begin{cases} r_t = (1, 2t, 0) \\ r_z = (0,0,1) \end{cases}$$

$$r_t \times r_z = (2t, 0, 0) \rightsquigarrow |r_t \times r_z| = 2t$$

$$\iint_S f\, dS = \iint_R t\, dA = \int_0^2 \int_0^{t^2+t^4} t\, dz\, dt = \cdots$$

<u>Surface integral of a vector field</u>



$$\underbrace{\iint_S \vec{F} \cdot d\vec{S}}_{} = \text{flux of } F \text{ across surface } S$$

$$= \iint_S \vec{F}(r(u,v)) \cdot \vec{n}\, dS$$

$$\vec{n} = \pm \frac{r_u \times r_v}{|r_u \times r_v|}$$

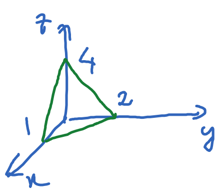
$$dS = |r_u \times r_v|\, dA$$

$$\rightsquigarrow \iint_S \vec{F} \cdot d\vec{S} = \iint_R \vec{F}(r(u,v)) \cdot (\pm r_u \times r_v)\, dA$$

<u>Ex</u>

$$F(x,y,z) = (4x+z, -x-y, 2y+z)$$

S = triangle with vertices $(1,0,0)$, $(0,2,0)$, $(0,0,4)$, facing downward

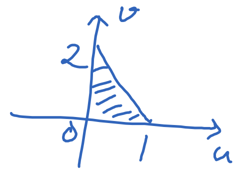


Parametrization of S:

$$\frac{x}{1}+\frac{y}{2}+\frac{z}{4}=1$$

$$\rightsquigarrow \quad 4x+2y+z=4$$

$$\rightsquigarrow \quad \begin{cases} x=u \\ y=v \\ z=4-4u-2v \end{cases} \qquad (u,v)\in R$$



$v=-2u+2$

$$\left.\begin{aligned} r_u &= (1,0,-4) \\ r_v &= (0,1,-2) \end{aligned}\right\} \quad r_u\times r_v=(4,2,1)$$

$\rightsquigarrow$ normal vector pointing downward is $-r_u\times r_v=(-4,-2,-1)$.

$$F(r(u,v))=F(u,v,4-4u-2v)=(4-2v,-u-v,4-4u)$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_R (4-2v,-u-v,4-4u)\cdot(-4,-2,-1)\,dA$$

$$=-\iint_R (16-8v-2u-2v+4-4u)\,dA$$

$$=-\iint_R (20-6u-10v)\,dA$$

$$=-\int_0^1\int_0^{-2u+2}(20-6u-10v)\,dv\,du=-\frac{34}{3}$$